Random walk for interacting particles on a Sierpínski gasket

Aniket Bhattacharya
Department of Chemistry, The Pennsylvania State University, 152 Davey Laboratory, University Park, Pennsylvania 16802
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The random motion for an arbitrary number of particles is numerically studied on a finitely ramified Sierpínski gasket with the restriction that no two particles can occupy the same site. We find that the long time behavior of a "tagged particle" in the presence of others is the same as that of a noninteracting one, i.e., $\langle r^2(t)\rangle \sim t^{2/d_w}$, where $d_w=2.322$ is the fractal dimensionality of the path of a single random walker in the gasket. This is contrary to the results for a one-dimensional (1D) chain where the presence of the interaction markedly alters the motion of the walkers. However, the collective displacement for all the particles undergoes *normal* diffusion, as on a 1D chain, with a characteristic length $\langle |\Delta R|^2\rangle \sim 1-\rho$, where ρ is the density of particles per site. We have shown that this should be true in general for all random walks in regular and self-similar fractal lattices.

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It is well known for a regular lattice in integer dimensions the mean-square displacement of a random walker is proportional to time (Fick's Law) [1]. More generally, it is expressed as

$$\langle r^2(t)\rangle \sim t^{2/d_w} , \tag{1}$$

where d_w is the dimension for the random walk. For a regular lattice $d_w=2$, hence $\langle r(t)^2\rangle \sim t$. For lattices with fractal dimensions the diffusion becomes anomalous, because d_w takes values greater than 2. While the random walk for a single particle in fractal dimensions is well studied, almost nothing is known when many interacting particles are present. For the case of a one-dimensional (1D) chain, Richards [2] numerically studied the hopping motion of an arbitrary number of random walkers, forbidding their double occupancy on a given site. It turned out that the introduction of an interaction among particles through this excluded volume effect drastically altered the random motion of a tagged particle in the presence of others. Instead of $\langle r^2(t)\rangle \sim t$, the behavior of a tagged particle becomes $\langle r^2(t)\rangle \sim t^{1/2}$, but the displacement for all the particles $\langle R^2(t)\rangle$ still maintains the linear behavior in time. Although the large time behavior can be derived analytically from a multiple scattering equation [3,4], a very simple but transparent derivation was given by Alexander and Pincus [5], where they argued that the long-time behavior is dominated by density fluctuations leading to the $t^{1/2}$ dependence of the mean-square deviation. It's also believed that in two dimensions the diffusion of a single particle in the presence of others is normal [6], thereby motivating further studies of the random walk problem in a fractal dimension which lies between 1 and 2.

In this paper we study the random walk of interacting particles by disallowing their double occupancies on a finitely ramified two-dimensional (2D) Sierpínski gasket. A Sierpínski gasket is a self-similar fractal. Its study has become very popular since a decade ago when it was suggested that it could be viewed as a backbone lattice of a percolating cluster [8]. Many linear problems have already been studied using *exact* renormalization group techniques [9]. Unfortunately, an exact solution is difficult, if not impossible, for many interacting particles executing random walks on the gasket. Here we resort to numerical methods to study a few properties of many interacting particles. Our results show that in a Sierpínski gasket, the random walk of a tagged particle is very different compared to what happens in one dimension. At long times an interacting particle begins to behave like a noninteracting one. We have also established a general property of the collective displacement of all the particles; it is always normal on a regular, or on a self-similar fractal lattice.

Before proceeding further, let us recapitulate a few things about a Sierpínski gasket. As shown in Fig. 1, a Sierpínski gasket [7] is generated starting with a triangle, breaking it into four equal smaller triangles, and then taking away the inner one. The process is repeated for each triangle, so generated to produce smaller triangles at the next stage of iteration. Hence its fractal dimension d_f is given by $2^{d_f}=3$ or $d_f=\ln 3/\ln 2=1.585$. The total number of triangles after N iterations is 3^N, the number of sites being $(3^{(N+1)}+3)/2$. The linear dimension of the gasket is given by 2^N. The spectral dimension d_s of a

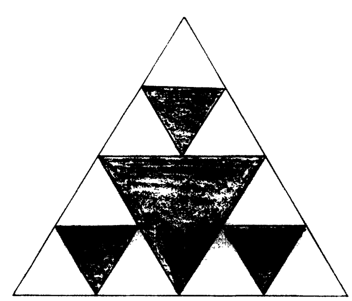


FIG. 1. Geometry of a Sierpínski gasket. The particles are not allowed to move into the shaded regions.

fractal lattice is given by $d_s = 2d_f/d_w$ [10]. Along with Eq. (1), a single random walker in a 2D Sierpínski gasket also satisfies the following equation:

$$S_0(t) \sim t^{d_s/2} , \tag{2}$$

where $S_0(t)$ is the number of distinct sites visited. The values of d_w and d_s are given by 2.322 and 1.365, respectively.

In our work, we have chosen the value of N to be 7. This gives a gasket of length 128 and with a total number of 3282 sites. With a gasket of this size it is possible to study random walks up to 600 time steps without encountering boundary effects. We have checked our results for $N=5$ with that $N=7$ at a fixed density of particles, up to 100 time steps. The convergence is extremely good. The sites on a gasket are labeled with two integers n and m, such that the Cartesian components of the vector $\mathbf{r}(n,m)$ are given by $(n+m/2)$ and $(\sqrt{3}/2)m$, respectively. We apply the standard method of tossing a coin to study the random walk. Initially, the particles are distributed randomly on different sites of the gasket. A nearest neighbor of a given particle is chosen by generating a random number. If the neighboring site is empty then the move is accepted, otherwise the particle stays in its original site.

For the sake of clarity, a "tagged particle" will refer to the motion of a particular particle in the presence of others with the excluded volume interaction, whereas a "single particle" would refer to a particle without any other particles on the gasket. The quantities that have been calculated in this paper are as follows: (i) the root-mean-square distance

$$\langle r^2(t) \rangle = \frac{1}{n_p} \left\langle \sum_{t=1}^{n_p} r_i^2(t) \right\rangle , \tag{3}$$

where n_p represent the total number of particles; (ii) the number of distinct sites $S(t)$ visited by a tagged particle; and (iii) the root-mean-square displacement of the collective coordinate $\mathbf{R} = \mathbf{r}_1 + \mathbf{r}_2 + \cdots + \mathbf{r}_{n_p}$,

$$\langle R^2(t) \rangle = \langle |\mathbf{r}_1 + \mathbf{r}_2 + \cdots + \mathbf{r}_{n_p}|^2 \rangle . \tag{4}$$

For a single particle executing a random walk the root-mean-square displacement and the number of distinct sites visited are denoted by $\langle r_0^2(t) \rangle$ and $S_0(t)$, respectively. The density of the particles is chosen to be 0.5 to facilitate faster convergence. In Fig. 2(a) we have plotted $\langle r_0^2(t) \rangle, \langle r^2(t) \rangle, \langle R^2(t) \rangle$ as a function of time on a log-log plot. At long time the slopes of the two lines (the top and the bottom one) become parallel. It is easier to see it if one plots $\ln(\langle r^2(t) \rangle) - \ln(\langle r_0^2(t) \rangle) \sim \ln(t)$, which is shown in Fig. 2(b). If the behavior of $\langle r^2(t) \rangle$ and $\langle r_0^2(t) \rangle$ have different powers of time, $\langle r^2(t) \rangle \sim t^{\alpha}$ and $\langle r_0^2(t) \rangle \sim t^{\alpha_0}$ ($\alpha_0 = 2/d_w = 0.8613$), say, then

$$\ln(\langle r_0^2(t) \rangle) - \ln(\langle r^2(t) \rangle) \sim (\alpha_0 - \alpha) \ln(t) . \tag{5}$$

The fact that, at long time, the slope becomes zero implies that the exponent α eventually approaches α_0. Hence the motion of the tagged particle at long time is characterized by the same exponent as that of a single particle. In Fig. 3(a) we have shown the number of distinct sites as a function of time for a tagged particle (bottom) and a single particle (top), respectively. Figure 3(b) shows the difference $\ln[S(t)] - \ln[S_0(t)] \sim \ln(t)$. It also shows similar behavior. If $S(t) \sim t^{\beta}$, then in the long time limit β asymptotically takes the value d_f/d_w, the same as that of a single particle.

On the contrary, the motion of the sum of the coordinates of all the particles $\mathbf{R}$ is the same as that for the 1D lattice. Figure 2(a) also shows the temporal behavior of $\ln\langle R^2(t) \rangle$ (the middle one). Notice that the slope for $\langle R^2(t) \rangle$ increases continuously. If we assume that $\langle R^2(t) \sim t^{\gamma}$, then $\ln\langle r_0^2(t) \rangle - \ln\langle R^2(t) \rangle \sim (\alpha_0 - \gamma) \ln(t)$. The bottom curve of Fig. 2(b) shows the behavior of this expression as a function of $\ln(t)$. Notice that unlike the

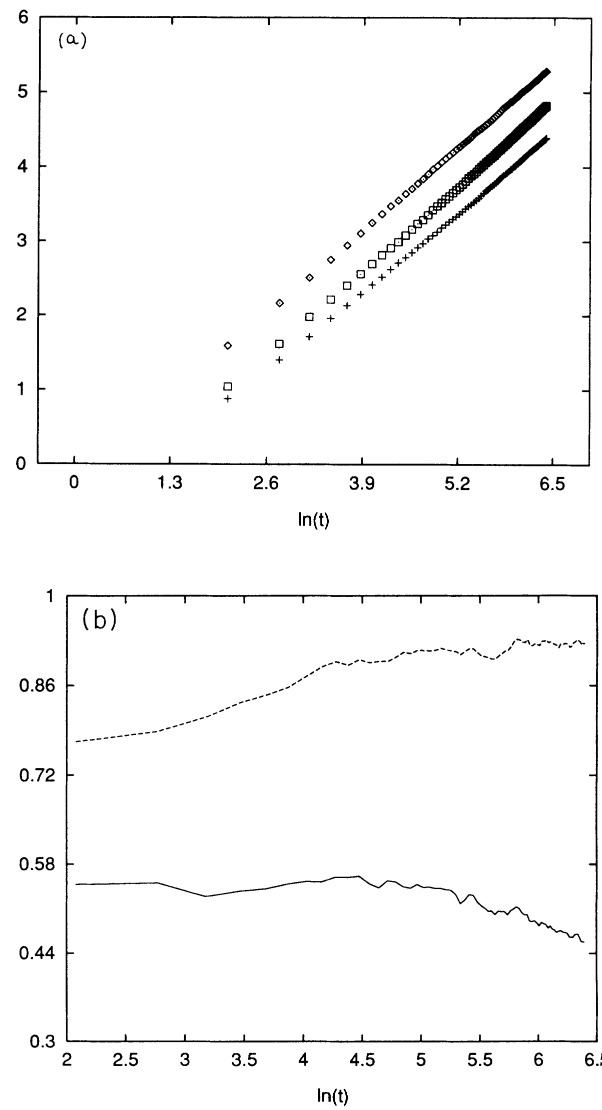
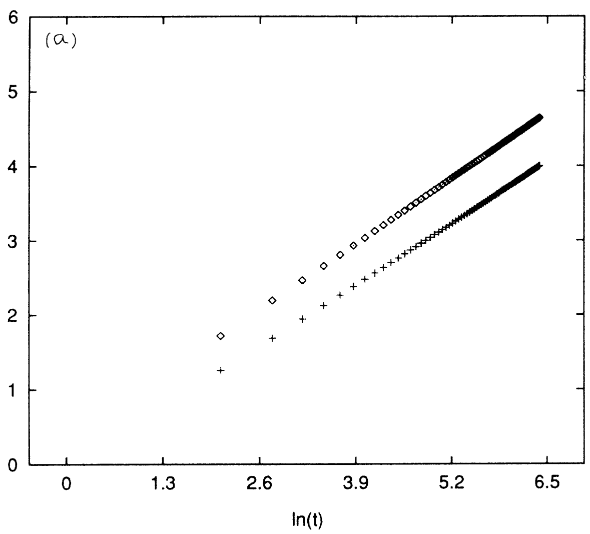


FIG. 2. (a) $\langle r_0^2(t) \rangle$ (top), $\langle r^2(t) \rangle$ (bottom), and $\langle R^2(t) \rangle$ (middle) as a function of t on a log-log scale. (b) $\ln\langle r_0^2(t) \rangle - \ln\langle r^2(t) \rangle$ (top) and $\ln\langle r_0^2(t) \rangle - \ln\langle R^2(t) \rangle$ (bottom) as a function of $\ln(t)$.

top curve it decreases as a function of time, implying that the coefficient $\gamma > \alpha_0$. We argue that eventually $\langle R^2(t) \rangle \sim t$ (i.e., $\gamma \rightarrow 1$) and this result is true in general, for all self-similar fractal and regular lattices. Let us establish it in case of a 2D Sierpínski gasket first. The proof follows from the fact that if the coordinates $(n_1,m_1),(n_2,m_2),\cdots,(n_{n_p},m_{n_p})$, representing the position of random walkers are located on the fractal lattice, it is easy to check that the vector,

$$\mathbf{R}=(n_1+n_2+\cdots+n_{n_p};m_1+m_2+\cdots+m_{n_p}) , \qquad (6)$$

is located on the underlying triangular lattice but not necessarily on the fractal lattice. One can also see it pictorially from Fig. 1. This is because the vector **R** might very well fall into the shaded region of Fig. 1, where individual particles are not allowed. Therefore the vector **R**



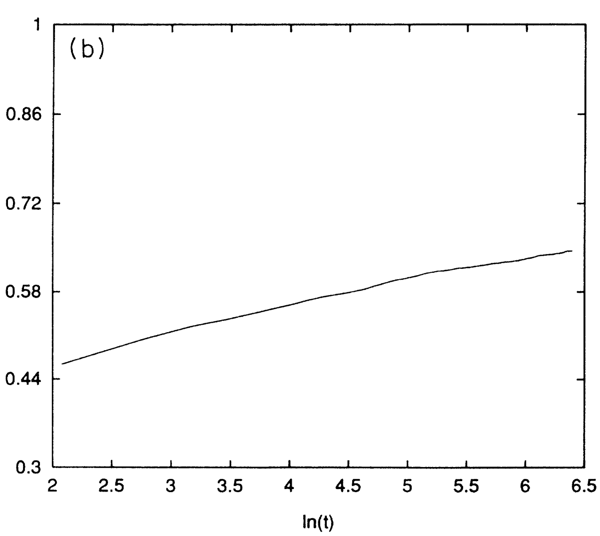


FIG. 3. (a) $\ln\langle S_0(t)\rangle$ (top) and $\ln\langle S(t)\rangle$ bottom as a function of $\ln(t)$. (b) $\ln\langle S_0(t)\rangle - \ln\langle S(t)\rangle$ as a function of $\ln(t)$.

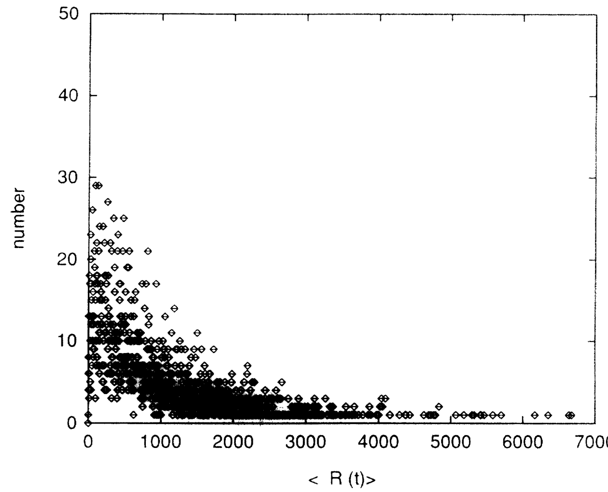


FIG. 4. A typical spread of $|\Delta\mathbf{R}|^2$ as a function number of number of iteration after being sorted in an increasing order of magnitude.

executes a random motion on a triangular lattice. Hence the motion is *normal,* and therefore *linear* in time. One notices that nothing special about a Sierpínski gasket is needed for the proof. Therefore the result should be valid in general for all self-similar fractals.

We have also explored properties of the vector **R** numerically, which would be consistent if the above argument were true. For example, let us consider the following quantity:

$$\langle |\Delta\mathbf{R}|^2 \rangle = \langle |\mathbf{R}(t+1)-\mathbf{R}(t)|^2 \rangle . \qquad (7)$$

Since the vector **R** is a sum of random moves for all the particles, and we just proved that it is spanning a regular triangular lattice, the average jump would than only depend on the available volume fraction $1-\rho$, where ρ is the average number of particles per site. Hence it is expected that $\langle |\Delta\mathbf{R}|^2 \rangle$ would show a peak at this characteristic distance, depending on the density and decrease rapidly beyond that. Also, since any two successive moves are uncorrelated, $\langle |\Delta\mathbf{R}|^2 \rangle$ would be invariant with respect to time. In Fig. 4 we have shown $|\Delta\mathbf{R}|^2$ for different realizations after sorting them in increasing order of their magnitudes. We have checked numerically at several different times that $\langle |\Delta\mathbf{R}|^2 \rangle \sim 1-\rho$. This is consistent at the two extreme limits. In the first case when there is only one particle present, in the thermodynamic limit $\langle |\Delta\mathbf{R}|^2 \rangle$ should be exactly unity; in the other extreme, when the number of particles approaches the number of sites $\langle |\Delta\mathbf{R}|^2 \rangle$ should approach zero.

In summary, we have numerically investigated the random walk of interacting particles on a finitely ramified Sierpínski gasket. The behavior of a tagged particle asymptotically becomes the same as that of a single particle. Similar behavior is expected for other lattices with fractal dimensions greater than 1. In other words, we believe that one dimension is special. We have also shown that the collective displacement of all the particles is as in normal diffusion and argued that this should be true for all regular and self-similar fractal lattices.

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